JOURNAL OF APPROXIMATION THEORY 25, 56–64 (1979)

Nonlinear Chebyshev Approximations Having Restricted Ranges

RYSZARD SMARZEWSKI

Department of Numerical Methods, M. Curie–Sklodowska University, 20-031 Lublin, Poland

Communicated by G. Meinardus

Received June 20, 1977

1. INTRODUCTION

Let $C(X)$ denote the space of real continuous functions defined on a compact Hausdorff space X endowed with the uniform norm

$$\|f\| = \sup\{|f(x)|: x \in X\}.$$

For any function f defined on X, denote

$$Z_f = \{x \in X: f(x) = 0\}$$

and

$$M_f = \{x \in X: |f(x)| = \|f\|\}.$$

Let l, u be given functions from X into the extended real line $[-\infty, \infty]$, with $l < u$. Additionally, let G be a proper subset of $C(X)$.

DEFINITION 1. $g \in G$ is said to be a best approximation to $f \in C(X)$ in G if

$$\|f - g\| \leqslant \|f - h\|$$

for all $h \in G$.

Denote

$$G^* = \{g \in G: l \leqslant g \leqslant u\}.$$

DEFINITION 2. A best approximation $g \in G^*$ to f in G^* is said to be a best restricted approximation.

Best restricted approximations have been investigated recently in wide variety of papers (see [6] for references).

DEFINITION 3. A subset G of $C(X)$ has the weak betweenness property if for any distinct elements g and h of G and for every nonempty closed

0021-9045/79/010056–09$02.00/0

subset D of X such that $\min\{|h(x) - g(x)| : x \in D\} > 0$ there exists a sequence $\{g_i\}$ in G such that

(i) $\lim_{i\to\infty} \| g - g_i \| = 0$ and

(ii) $\min\{[h(x) - g_i(x)][g_i(x) - g(x)]: x \in D\} > 0$ for all i.

As we noted in [4], subsets having the weak betweenness property include, e.g., subsets with the betweenness property, asymptotically convex and having a degree. In this paper we shall give a characterization theorem of Kolmogorov type for the best restricted approximation by the elements of a subset G having the weak betweenness property. Additionally, we shall obtain some converse theorem, i.e., if the necessity of the characterization theorem is true for all $f \in C(X)$ then some subset of G has the weak betweenness property. These two results will be obtained under additional assumptions that l is an upper semicontinuous function and u is a lower semicontinuous function into the extended real line. We note that our assumptions about the restrictions l and u are different from Dunham's assumptions in [1]. However, if the functions l and u satisfy five restrictions (i)–(v) in [5, p. 242] then, of course, l and u are upper and lower semicontinuous functions, respectively.

2. Main Results

At first, we give a sufficient condition for $g \in G^*$ to be a best restricted approximation for f in G^*. This does not require any assumptions about the structure of the set G and the properties of the restrictions l and u.

THEOREM 1. *A sufficient condition for $g \in G^*$ to be the best restricted approximation to $f \in C(X)$ in G^* is that the inequality*

$$\max\{[g(x) - h(x)] \operatorname{sign}[f(x) - g(x)]: x \in M_{f-g}\} \geqslant 0 \tag{1}$$

be satisfied for every $h \in G^*$.

Proof. From the continuity of functions f, g, h on X and $(g - h)$ $\operatorname{sign}(f - g)$ on M_{f-g}, and the compactness of X and M_{f-g} we have, by (1)

$$\begin{aligned}\|f - g\| &\leqslant \|f - g\| + \max\{[g(x) - h(x)] \operatorname{sign}[f(x) - g(x)]: x \in M_{f-g}\} \\ &= [f(z) - g(z)] \operatorname{sign}[f(z) - g(z)] + [g(z) - h(z)] \operatorname{sign}[f(z) - g(z)] \\ &= [f(z) - h(z)] \operatorname{sign}[f(z) - g(z)] \leqslant \|f - h\|\end{aligned}$$

for all $h \in G^*$, where $z \in M_{f-g} \cap M_{(g-h)\operatorname{sign}(f-g)}$ and the domain of the function $(g-h)\operatorname{sign}(f-g)$ is restricted to M_{f-g}. This completes the proof. ▮

Let us define now the subset G_g of G by

$$G_g = \{h \in G\colon h(x) > g(x) \text{ and } h(y) < g(y) \text{ for each } x \in Z_{l-g} \text{ and } y \in Z_{u-g}\},$$

where g is a fixed element of G^*. In the following we shall use the well-known properties of lower and upper semicontinuous functions in the extended real line $[-\infty, \infty]$ (see, for example, [2, pp. 73–77]). If G has the weak betweenness property then the following theorem is true:

THEOREM 2. *Let l and u be, respectively, upper and lower semicontinuous functions into the extended real line. Then a necessary condition for $g \in G^*$ to be the best restricted approximation to $f \in C(X)$ in G^* is that inequality* (1) *be satisfied for all $h \in G_g$.*

Proof. Let us suppose, on the contrary, that there exists an element $h \in G_g$ such that

$$\max\{[g(x) - h(x)] \operatorname{sign}[f(x) - g(x)]\colon x \in M_{f-g}\} < 0. \tag{2}$$

Since $l \leqslant g \leqslant u$, then

$$Z_{l-g} = \{x \in X\colon l(x) - g(x) \geqslant 0\}$$

and

$$Z_{u-g} = \{x \in X\colon u(x) - g(x) \leqslant 0\}.$$

This and the upper semicontinuity of l and the lower semicontinuity of u imply that the sets Z_{l-g} and Z_{u-g} are closed.

Thus, from $h \in G_g$ it follows

$$\min\{h(x) - g(x)\colon x \in Z_{l-g}\} > 0 \tag{3}$$

and

$$\min\{g(x) - h(x)\colon x \in Z_{u-g}\} > 0. \tag{4}$$

Additionally, from (2) we have

$$\begin{aligned} &\min\{|g(x) - h(x)|\colon x \in M_{f-g}\} \\ &\qquad \geqslant \min\{[g(x) - h(x)] \operatorname{sign}[f(x) - g(x)]\colon x \in M_{f-g}\} > 0. \end{aligned} \tag{5}$$

Let us define the closed set D by

$$D = M_{f-g} \cup Z_{l-g} \cup Z_{u-g}.$$

From (3)–(5) we have

$$\min\{|g(x) - h(x)|\colon x \in D\} > 0.$$

Since G has the weak betweenness property, there exists the sequence $\{g_i\}$ in G such that

$$\lim_{i\to\infty} \| g - g_i \| = 0 \tag{6}$$

and

$$\min\{[h(x) - g_i(x)][g_i(x) - g(x)]: x \in D\} > 0. \tag{7}$$

Referring to (6), let n be an integer such that $\| g - g_i \| < \| f - g \|$ for all $i > n$.

Hence, from (2), (5), and (7) we obtain

$$\min\{| g_i(x) - g(x)|: x \in M_{f-g}\} > 0$$

and

$$\operatorname{sign}[f(x) - g(x)] = \operatorname{sign}[f(x) - g_i(x)] = \operatorname{sign}[g_i(x) - g(x)]$$

for every $i > n$ and $x \in M_{f-g}$. This implies that

$$|f(x) - g_i(x)| = |f(x) - g(x)| - |g_i(x) - g(x)| < \| f - g \| \tag{8}$$

for these i and x.

If $M_{f-g} = X$ then the proof is completed. Otherwise, the continuity arguments imply that there exists an open set $N \supset M_{f-g}$ such that (8) holds for all $i > n$ and $x \in N$. Since the set $V = X \backslash N$ is compact and $M_{f-g} \cap V \neq \varnothing$ then

$$\delta_1 = \max\{|f(x) - g(x)|: x \in V\} < \| f - g \|.$$

Let $n_1 \geqslant n$ be so selected that the inequality

$$\| g - g_i \| < \| f - g \| - \delta_1$$

hold for all $i > n_1$. Therefore, we have

$$\begin{aligned} |f(x) - g_i(x)| &\leqslant |f(x) - g(x)| + |g(x) - g_i(x)| \\ &< \delta_1 + \| f - g \| - \delta_1 = \| f - g \| \end{aligned}$$

for all $i > n_1$ and $x \in V$. From this inequality and from that for $x \in N$ it follows that

$$\| f - g_i \| < \| f - g \| \tag{9}$$

for all $i > n_1$.

Now, for the completion of the proof, it is sufficient to show that there exists at least one index $i > n_1$ such that $g_i \in G^*$. To this purpose define

$$\delta_2 = \min\{u(x) - l(x): x \in X\}.$$

From the compactness of X, the inequality $l < u$, and the lower semicontinuity of $(u - l)$ it follows that $\delta_2 > 0$. Referring to (6), we select an integer $n_2 \geqslant n_1$ such that

$$\| g - g_i \| < \delta_2$$

for all $i > n_2$. Because $h \in G_g$, using (7) we have for each $x \in Z_{l-g}$ and $y \in Z_{u-g}$

$$g_i(x) - l(x) = g_i(x) - g(x) > 0$$

and

$$u(y) - g_i(y) > 0.$$

Additionally, for these x, y and $i > n_2$ we obtain

$$u(x) - g_i(x) = u(x) - g(x) - [g_i(x) - g(x)] \geqslant \delta_2 - \| g_i - g \| > 0$$

and

$$g_i(y) - l(y) \geqslant \delta_2 - \| g - g_i \| > 0.$$

Therefore, we have established that

$$l(x) < g_i(x) < u(x) \tag{10}$$

for each $x \in Z_{l-g} \cup Z_{u-g}$ and $i > n_2$.

If $X = Z_{l-g} \cup Z_{u-g}$ then the proof is completed. Otherwise, from the upper semicontinuity of l and lower semicontinuity of u it follows that there exists an open set $N \supset Z_{l-g} \cup Z_{u-g}$ such that inequality (10) holds for each $x \in N$. Let us set $V = X \backslash N$. Because V is a closed set and $V \cap (Z_{l-g} \cup Z_{u-g}) = \varnothing$ then by the lower semicontinuity of functions $u - g$ and $g - l$ we have

$$\delta_3 = \min\{u(x) - g(x) \colon x \in V\} > 0$$

and

$$\delta_4 = \min\{g(x) - l(x) \colon x \in V\} > 0.$$

Let $n_3 \geqslant n_2$ be so chosen that

$$\| g - g_i \| < \min(\delta_3, \delta_4)$$

for each $i > n_3$. Thus, for each $x \in V$ and $i > n_3$ we have

$$u(x) - g_i(x) = u(x) - g(x) + [g(x) - g_i(x)] > \delta_3 - \| g - g_i \| > 0$$

and

$$g_i(x) - l(x) > \delta_4 - \| g - g_i \| > 0.$$

Combining these two inequalities with that for $x \in N$ we conclude from (9) that every function g_i lies in G^* for $i > n_3$ and is a better restricted approximation to f than g. This completes the proof. ∎

Note that, in general, the set G_g does not contain the set G^*. Therefore, the sufficient condition for $g \in G^*$ to be a best restricted approximation in G^* is not a necessary condition.

Unfortunately, the following simple example shows that the set G_g in Theorem 2 can not be changed on the set $\bar{G}_g$ defined by

$$\bar{G}_g = \{h \in G: h(x) \geqslant g(x) \text{ and } h(y) \leqslant g(y) \text{ for each } x \in Z_{l-g} \text{ and } y \in Z_{u-g}\}.$$

EXAMPLE 1. Let $X = [-1, 1]$, $G = \{\alpha x: \alpha \in R\}$, $l(x) = -\infty$, $u(x) = x^2$, and $f(x) = 1 - x$. Then G^* contains only the zero function $g = 0$, which is the best restricted approximation to f. Additionally, $G_g = \varnothing$, $\bar{G}_g = G$, and $M_{f-g} = \{-1\}$. It is obvious, that inequality (1) does not hold for $h(x) = x \in \bar{G}_g$.

However, Example 1 does not answer the interesting question: Whether the set G_g may be changed on $G_g \cup (\bar{G}_g \cap G^*) = G_g \cup G^*$. At present, we do not know whether this is true or not. Therefore, the problem whether necessary and sufficient conditions exist for $g \in G^*$ to be the best restricted approximation in G^* is left open. The answer to this question is yes, particularly when G has the betweenness property. This follows easily from the fact that G^* has also the betweenness property.

DEFINITION 4. Let the two restriction functions $l < u$ be given. If B and V are closed subsets of X such that $B \cap V = \varnothing$ then the following two restrictions r and s defined by

$$\begin{aligned} r(x) &= -\infty, & x &\in X \backslash B, \\ &= l(x), & x &\in B, \end{aligned}$$

and

$$\begin{aligned} s(x) &= \infty, & x &\in X \backslash V, \\ &= u(x), & x &\in V, \end{aligned}$$

are called admissible restrictions.

Note that r and s are, respectively, upper and lower semicontinuous functions, if l and u are such ones, too. In the following theorem we shall additionally assume that X is a space with the metric $|\cdot|$.

THEOREM 3. *Let restrictions l and u be as in Theorem 2. If Theorem 2 holds for each $f \in C(X)$ and all admissible restrictions to l and u then the set $p \cup G_p$ has the weak betweenness property for each $p \in G^*$.*

Proof. Let us suppose that g and h are two distinct elements in $p \cup G_p$ and D is a nonempty closed subset of X such that

$$\delta_1 = \min\{|g(x) - h(x)|: x \in D\} > 0. \tag{11}$$

Let λ_i be a decreasing sequence of positive numbers convergent to zero. To prove this theorem it sufficient to show that there exists a sequence $\{g_i\}$ in $p \cup G_p$ such that

$$\| g - g_i \| < \lambda_i \tag{12}$$

and

$$\min\{[h(x) - g_i(x)][g_i(x) - g(x)]: x \in D\} > 0 \tag{13}$$

for all integers i.

At first, suppose that $g = p \in G^*$ and $h \in G_g$. Define the function $f_1 \in C(X)$ by

$$f_1(x) = g(x) + \epsilon_1 \frac{\rho(x, Z_{g-h})}{\rho(x, Z_{g-h}) + \rho(x, B)} \operatorname{sign}[h(x) - g(x)],$$

where

$$0 < \epsilon_1 < \tfrac{1}{2} \min(\lambda_1, \delta_1), \qquad B = D \cup Z_{l-g} \cup Z_{u-g},$$

and

$$\rho(x, E) = 1 \quad , \quad \text{if } E = \varnothing, \qquad = \inf\{| x - e |: e \in E\}, \quad \text{otherwise.}$$

Because

$$\max\{[g(x) - h(x)] \operatorname{sign}[f_1(x) - g(x)]: x \in B\} = - \min\{| h(x) - g(x)|: x \in B\} = -\delta_1 < 0$$

and $B = M_{f_1-g}$, from Theorem 2 it follows that g is not the best restricted approximation to f_1 in G, i.e., there exists the function $g_1 \in G^*$ such that

$$\| f_1 - g_1 \| < \| f_1 - g \| = \epsilon_1 .$$

Hence, we obtain

$$\| g - g_1 \| \leqslant \| f_1 - g_1 \| + \| f_1 - g \| < \lambda_1$$

which establishes (12) for $i = 1$. Additionally, since $| f_1(x) - g_1(x)| < | f_1(x) - g(x)|$ for all $x \in B = M_{f_1-g}$ then we have

$$\operatorname{sign}[g_1(x) - g(x)] = \operatorname{sign}[f_1(x) - g(x)] = \operatorname{sign}[h(x) - g(x)]$$

for all $x \in B$. Hence, it follows that

$$\begin{aligned}
&\min\{[h(x) - g_1(x)][g_1(x) - g(x)]: x \in B\} \\
&\quad = \min\{| f_1(x) - g(x) - [f_1(x) - g_1(x)]| (| h(x) - g(x)| - | g_1(x) - g(x)|) \\
&\qquad : x \in B\} \\
&\quad \geqslant \min\{(\epsilon_1 - | f_1(x) - g_1(x)|)(\delta_1 - \| f_1 - g_1 \| - \| f_1 - g \|): x \in B\} \\
&\quad \geqslant (\epsilon_1 - \| f_1 - g_1 \|)^2 > 0
\end{aligned}$$

which establishes (13) for $i = 1$. Denote $\delta_2 = \min\{|g_1(x) - g(x)|: x \in B\} > 0$. Now, from this last inequality and from the fact that $h \in G_g$ we have for each $x \in Z_{l-g}$ and $y \in Z_{u-g}$ that

$$g_1(x) - l(x) = g_1(x) - g(x) > 0$$

and

$$u(y) - g_1(y) > 0,$$

i.e., g_1 lies in G_g. Finally, replacing g_{i-2} ($g_0 = h$) by g_{i-1}, λ_{i-1} by λ_i, δ_{i-1} by δ_i, and ϵ_{i-1} by ϵ_i we may analogously construct by induction the functions f_i, $i = 2, 3,\ldots$ such that $B = M_{f_i - g}$ and that g are not the best restricted approximation to f_i. Additionally, denoting the better restricted approximation to f_i by g_i we may prove that (12) and (13) are satisfied and that $g_i \in G_g$. This completes the proof in case $g = p$.

Secondly, suppose that g and h lie in G_p and (11) holds. Define two functions r and s by

$$\begin{aligned} r(x) &= -\infty, \qquad & x \in X \backslash Z_{l-p}, \\ &= l(x), & x \in Z_{l-p}, \end{aligned}$$

and

$$\begin{aligned} s(x) &= \infty, \qquad & x \in X \backslash Z_{u-p}, \\ &= u(x), & x \in Z_{u-p}. \end{aligned}$$

Obviously, r and s are admissible restrictions. Let us denote

$$H^* = \{u \in G: r \leqslant u \leqslant s\}$$

and

$$H_g = \{u \in G: u(x) > r(x) \text{ and } u(y) < s(y) \text{ for each } x \in Z_{r-g} \text{ and } y \in Z_{s-g}\}.$$

We immediately have $g \in H^*$ and $h \in H_g = G_g$. Therefore changing G^* on H^* and taking into consideration what has been said about the case $g = p$ we may prove the existence of the sequence $\{g_i\}$ in H_g such that (12) and (13) hold. This completes the proof. ∎

3. Concluding Remarks

Now, let us briefly consider the best restricted approximation by elements of the set

$$G^0 = \{h \in G: l < h < u\}.$$

We may obtain, after trivial modifications of the proofs of Theorems 2 and 3, the following results. If G has the weak betweenness property, then the following theorem holds:

THEOREM 4. *Let l and u be, respectively, upper and lower semicontinuous functions into the extended real line. Then a necessary and sufficient condition for $g \in G^0$ to be the best restricted approximation to $f \in C(X)$ in G^0 is that inequality* (1) *be satisfied for all $h \in G^0$.*

THEOREM 5. *Let X be a metric space and l, u be as in the previous theorem. If Theorem 4 holds for each $f \in C(X)$ then G^0 has the weak betweenness property.*

These two results also follow immediately from [3] and from the fact that the subset G^0 of G has the weak betweenness property, if G has this property, too. Finally, we note that it is possible to generalize our results to the case when X is not compact and $C(X)$ is changed on the space $C_b(X)$ containing all real continuous and bounded functions defined on X. In this case the set M_{f-g} must be changed in Theorem 2 into

$$M_{f-g}(\epsilon) = \{ x \in X: |f(x) - g(x)| \geqslant \|f - g\| - \epsilon\},$$

where $\epsilon > 0$ is sufficiently small. Additionally, we ought to change the maximum on the supremum in all previous statements and assume that X in Theorem 3 is a normal space. These generalizations do not require new ideas in proofs in view of the considerations given in Section 2 and [4].

REFERENCES

1. C. B. DUNHAM, Chebyshev approximation with restricted ranges by families with the betweenness property, *J. Approximation Theory* **11** (1974), 254–259.
2. E. MCSHANE AND T. BOTTS, "Real Analysis," Van Nostrand, Princeton, N. J., 1959.
3. R. SMARZEWSKI, A note on characterization of family with weak betweenness property, submitted.
4. R. SMARZEWSKI, Some remarks on nonlinear Chebyshev approximation to functions defined on normal space, *J. Approximation Theory* **24** (1978), 169–175.
5. G. D. TAYLOR, Approximation by functions having restricted ranges, III, *SIAM J. Math. Anal.* **27** (1969), 241–248.
6. G. D. TAYLOR, Uniform approximation with side conditions, (1973), 495–503.